T-Shirt Explanation: Westmont's 28th Annual Mathematics Contest

A *perfect number* has the property that it equals the sum of its proper divisors. The number 28 is perfect because its proper divisors are 1, 2, 4, 7, and 14, and $1 + 2 + 4 + 7 + 14 = 28$. The T-shirt for Westmont's 28th annual mathematics contest (shown on the next page) illustrates this fact with lines connecting the proper divisors of 28 (displayed in blocks) that lead up to 28 in the top block. Inside the line-block pattern the T-shirt depicts a procedure that the Greek mathematician Euclid discovered for finding perfect numbers. In 300 BC he described it in Proposition IX.36 of his famous book, *The Elements*:

> If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.

In other words, begin by forming a sum of numbers starting with the number one. Each subsequent term of the sum is then set to equal twice the value of the preceding term. If at any point the sum is a prime number, then that sum multiplied by the last term of the sum will be a perfect number.

To illustrate, consider Euclid's "double proportion" sum with just three terms: $1+2+4$. The sum equals 7, which is a prime number, and the last term of the sum is the number 4. Thus, according to Euclid, the product $(7)(4) = 28$ should be perfect, which it is!

The sum isn't always prime, but if it is it is called a *Mersenne prime* in honor of Marin Mersenne (1588–1648), a French mathematician, theologian, philosopher, and music theorist. If there are n terms in the sum it can be expressed as $1 + 2 + 4 + \cdots + 2^{n-1}$, and has the compact representation

$$1 + 2 + 4 + \cdots + 2^{n-1} = \sum_{k=0}^{n-1} 2^k,$$

which is why the T-shirt has "prime" written sideways beside the sum just below the number 28.

The last term of the above sum is 2^{n-1}, and the next page shows that the sum itself equals $2^n - 1$. Thus, the portion of the T-shirt interior to the line-block pattern illustrates Euclid's discovery:

> If $\sum\limits_{k=0}^{n-1} 2^k$ is prime (upper part), then the product $(2^n - 1)\left(2^{n-1}\right)$ is perfect (lower part).

The proof of this discovery is also on the next page.

Note that, with $n = 3$, we get—as before—the perfect number $(2^3 - 1)(2^{3-1}) = (7)(4) = 28$. The use of $n = 3$ to get 28 explains why the bottom part of the T-shirt reads "CO(N=3)TEST."

Incidentally, it was not until 1749 that the Swiss mathematician Leonhard Euler proved that all even perfect numbers must come from Euclid's formula. It is not known today if there are any odd perfect numbers, but if there is one, it must contain at least 300 digits, and have at least 29 different prime factors.

No one knows how many perfect numbers there are. If there are no odd perfect numbers, then there is a one-to-one correspondence between perfect numbers and Mersenne primes. As of January, 2016 there were only 49 known Mersenne primes, thus only 49 known perfect numbers. The 49th one (found in September, 2015) equals $(2^{74,207,281} - 1)(2^{74,207,280})$, and has 44,677,235 digits.

Mathematical Arguments

1. Show that, for $n \geq 1$, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$.

Explanation:

Let $S = \sum_{k=0}^{n-1} 2^k = 1 + 2 + 4 + \cdots + 2^{n-1}$. Then

$$2S = 2 + 4 + 8 + 16 \cdots + 2^{n-1} + 2^n; \tag{1}$$

$$S = 1 + 2 + 4 + 8 + \cdots + 2^{n-1}. \tag{2}$$

Subtracting the respective sides of equation (2) from equation (1) gives $S = 2^n - 1$.

2. Show that Euclid was right: if $\sum_{k=0}^{n-1} 2^k$ is prime, then the product $P = \left(\sum_{k=0}^{n-1} 2^k\right)\left(2^{n-1}\right)$ is perfect.

Explanation:

Consider the two components that make up P above. If the first one, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$, is prime, then its only divisors are 1 and $2^n - 1$. The second one, 2^{n-1}, has $1,\ 2,\ 4,\ \ldots,\ 2^{n-1}$ as divisors, and none of them (besides 1) is a divisor of the prime number $2^n - 1$. Thus, the divisors of the product $P = \left(\sum_{k=0}^{n-1} 2^k\right)\left(2^{n-1}\right) = \left(2^n - 1\right)\left(2^{n-1}\right)$ are

$$1,\ 2,\ 4,\ \ldots,\ 2^{n-1}, \quad 2^n - 1,\ 2(2^n - 1),\ 4(2^n - 1),\ \ldots,\ 2^{n-1}(2^n - 1).$$

The sum of these divisors is

$$\begin{aligned}
\left(1 + 2 + 4 + \cdots + 2^{n-1}\right) + \left(1 + 2 + 4 + \cdots + 2^{n-1}\right)\left(2^n - 1\right) &= \left(2^n - 1\right) + \left(2^n - 1\right)\left(2^n - 1\right) \\
&= \left(2^n - 1\right)\left(1 + (2^n - 1)\right) \\
&= \left(2^n - 1\right)\left(2^n\right) \\
&= 2\left(2^n - 1\right)\left(2^{n-1}\right) \\
&= 2P.
\end{aligned}$$

Thus, the sum of the divisors of P is $2P$. One of the divisors of P, however, is P itself, so the sum of the *proper* divisors of P is P. Therefore, P is a perfect number.